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OPTIMAL MATCHING OF LINEAR TIME-VARYING NETWORKS

Robert John Tracey

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Robert John Tracey, M.S.

Department of Electrical Engineering
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ABSTRACT

The calculus of variations is used to obtain the optimal load network for a linear time-varying n-port network excited at each port by arbitrary signals. The given network is represented using a state variable description; use of a matrix of Lagrange multipliers in the variational process then leads directly to the state variable description of the adjoint matching network. The choice of anticipative or non-anticipative load networks is considered. The general result is then specialized to obtain the usual condition required for matching time-invariant networks in the periodic steady state. Conditions which guarantee the existence of an optimal match are obtained from the second variation. Finally, the entire matching process is interpreted in terms of the lossy and lossless parts of the network response.

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I. INTRODUCTION

Previous investigations of the time-varying matching problem[1] have been carried out using a matrix of impulse responses to represent the given network in the time domain. This approach has several inherent difficulties. First, the matching networks which are obtained are always purely anticipative; second, non-zero initial conditions on the given network cannot be easily handled; third, problems arise when the matrix contains impulses or doublets. Use of a state variable description of the given network allows the impulses and doublets to be easily handled, and non-zero initial conditions can easily be taken into account. In addition, the resulting state variable description of the load does not demand a purely anticipative matching network.

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[1] Rohrer, R. A., "Optimal Matching of Linear Networks", Coordinated Science Laboratory, University of Illinois, Urbana, Illinois, Report R-187, Contract DA-36-039-AMC 02208(E), December 1963.

II. THE TIME-VARYING MATCHING PROBLEM

The general time-varying matching problem is depicted in Figure 1, where a given linear time-varying network M is shown excited at each port by arbitrary currents. The load network N is to be found such that the energy delivered to N by the arbitrary currents over the time interval [a,b] is a maximum. The given network M is described by the two state variable equations[1]

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{v}(t) \tag{1}$$

$$\underline{i}_M(t) = \underline{C}(t)\underline{x}(t) + \underline{D}(t)\underline{v}(t) + \underline{E}(t)\dot{\underline{v}}(t) \ , \tag{2}$$

where $\underline{x}(t)$ is the state vector of the given network[2]. It is important to note that this state vector only determines the stored energy associated with the proper part of the network response. Some of the energy-storing elements of the network are controlled directly by the voltage $\underline{v}(t)$, resulting in the improper impulse response represented by the matrix $\underline{E}(t)$. The effect of this energy storage does not appear in the state equation (1), but only in the output equation (2).

The energy delivered to the load, which is to be maximized for arbitrary source currents, is given by

$$e = \int_a^b \underline{v}^T(t)\underline{i}_N(t)\ dt \ , \tag{3}$$

which can be written in terms of the aribtrary source currents and the

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[1] Zadeh, L. A., and C. A. Desoer, Linear System Theory, McGraw-Hill, New York, 1963.

[2] Note that underlined capital letters denote matrices, and underlined lower case letters denote column vectors. Dot denotes differentiation with respect to time, and superscript T will be used to indicate transpose.

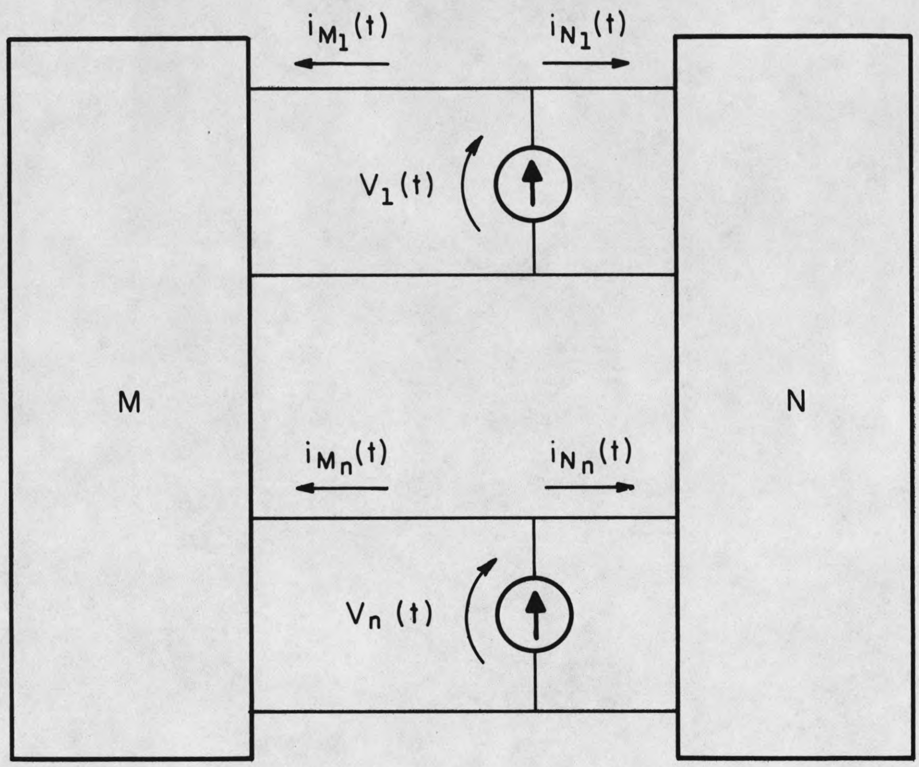

Figure 1. The Time-Varying Matching Problem

given network functions as

$$e = \int_a^b \underline{v}^T(t)[\underline{i}_S(t) - \underline{C}(t)\underline{x}(t) - \underline{D}(t)\underline{v}(t) - \underline{E}(t)\dot{\underline{v}}(t)]dt . \qquad (4)$$

This functional e depends not only on the variables $\underline{x}(t)$ and $\underline{v}(t)$ associated with the given network, but also on the arbitrary source current $\underline{i}_S(t)$. The functions $\underline{x}(t)$ and $\underline{v}(t)$ which yield a maximum value for the functional e can be found for each particular source current by means of the calculus of variations. Both $\underline{x}(t)$ and $\underline{v}(t)$ must be varied; however, these functions are not independent, but instead are related by equation (1). This constraint relation is included in the variational process by introducing a vector of Lagrange multipliers $\underline{\lambda}(t)$. Independent variations are then taken of the functional

$$e' = \int_a^b \left\{ \begin{array}{l} \underline{v}^T(t)[\underline{i}_S(t) - \underline{C}(t)\underline{x}(t) - \underline{D}(t)\underline{v}(t) - \underline{E}(t)\dot{\underline{v}}(t)] \\ - \underline{\lambda}^T(t)[\dot{\underline{x}}(t) - \underline{A}(t)\underline{x}(t) - \underline{B}(t)\underline{v}(t) \end{array} \right\} dt . \qquad (5)$$

Variation of $\underline{v}(t)$ (Appendix) yields the first necessary Euler equation

$$\begin{aligned} &\underline{i}_S(t) - \underline{C}(t)\underline{x}(t) - \underline{D}(t)\underline{v}(t) - \underline{E}(t)\dot{\underline{v}}(t) + \underline{B}^T(t)\underline{\lambda}(t) \\ &\qquad - \underline{D}^T(t)\underline{v}(t) + \dot{\underline{E}}^T(t)\underline{v}(t) + \underline{E}^T(t)\dot{\underline{v}}(t) = \underline{0} , \end{aligned} \qquad (6)$$

and variation of $\underline{x}(t)$ yields the second necessary Euler equation

$$\dot{\underline{\lambda}}(t) = -\underline{A}^T(t)\underline{\lambda}(t) + \underline{C}^T(t)\underline{v}(t) . \qquad (7)$$

Solution of these Euler equations together with the equation of constraint (1) will yield the desired optimal functions $\underline{x}(t)$ and $\underline{v}(t)$.

However, these are differential equations, and the functions $\underline{x}(t)$ and $\underline{v}(t)$ which (possibly) extremize the functional e (pending examination of the second variation) are only determined so far to within the arbitrariness of the initial conditions needed to specify a unique solution to the system of equations (1), (6), and (7). The appropriate initial conditions are found, for each particular source current, from the requirement that the conjuncts, which arise when terms containing derivatives of the variations are integrated by parts, (Appendix) must vanish:

$$[\delta \underline{x}^T(t)\underline{\lambda}(t)]_a^b = 0 \tag{8}$$

$$[\delta \underline{v}^T(t)\underline{E}^T(t)\underline{v}(t)]_a^b = 0 \ . \tag{9}$$

These two equations always provide exactly the required number of boundary conditions needed to specify a unique solution to equations (1), (6), and (7). The particular relations which are obtained from equations (8) and (9) depend on the manner in which boundary conditions are originally specified. (Equation (1) relates $\underline{v}(t)$ and $\underline{x}(t)$ only to within the arbitrariness of $\underline{x}(a)$.)

First consider the simplified case in which $\underline{E}(t) = \underline{0}$ and none of the components of $\underline{x}(t)$ are controlled directly by $\underline{i}_M(t)$ (i.e. no series inductances or shunt capacitances at the ports of the given network). Then a complete set of boundary conditions for equations (1), (6), and (7) must include 2n independent conditions, where n is the number of components of $\underline{x}(t)$. It is possible to specify the initial and final states $\underline{x}(a)$ and $\underline{x}(b)$ before the problem solution is undertaken; then $\delta\underline{x}(t)$ vanishes at each end point, and the conjunct 8 vanishes (equation (9) is satisfied by the assumption $\underline{E}(t) = \underline{0}$). Since 2n boundary conditions have been specified, both $\underline{x}(t)$ and $\underline{\lambda}(t)$ are uniquely determined. (When solving a system of 2n linear first order differential equations, it is not at all necessary to specify one boundary condition on each function. The only criterion that a set of boundary conditions must meet is that it lead to a set of 2n independent equations which can be solved for the coefficients in the differential equations.) More commonly, only the initial state $\underline{x}(a)$ is specified. Then $\underline{x}(b)$ can be specified at will, so $\delta\underline{x}(b)$ is arbitrary. Hence it must be required that $\underline{\lambda}(b) = \underline{0}$ in order to satisfy equation (8). In the most general situation, neither $\underline{x}(a)$ nor $\underline{x}(b)$ is specified; then it must be required that $\underline{\lambda}(a) = \underline{\lambda}(b) = \underline{0}$. The functions which satisfy these boundary conditions yield the greatest possible value for the functional e (provided that the sufficiency conditions are also met). In general, whenever any component of $\underline{x}(t)$ is not specified in the original formulation of the problem, it is required that the corresponding component of $\underline{\lambda}(t)$ be zero at the appropriate end point. In mathematical terms, equation (8) demands that $\underline{x}(t)$ and $\underline{\lambda}(t)$ satisfy adjoint boundary conditions.[3]

When shunt capacitances are present at the ports of the given network, $\underline{E}(t) \neq \underline{0}$. If the matrix $\underline{E}(t)$ is not symmetric (a non-reciprocal network), then the matrix $[\underline{E}(t) - \underline{E}^T(t)]$ is not zero. Exactly r additional boundary conditions must be specified, where r is the number of non-zero columns of $[\underline{E}(t) - \underline{E}^T(t)]$, for if a column of this matrix is zero, the corresponding component of $\dot{\underline{v}}(t)$ does not appear in equation (6). Because $\underline{x}(t)$ and $\underline{v}(t)$ are varied independently, the previous requirements on the boundary conditions

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[3] Lanczos, C., Linear Differential Operators, Van Nostrand, London, 1961. Note that the boundary conditions are adjoint if $\underline{x}^T(a)\underline{\lambda}(a) = \underline{x}^T(b)\underline{\lambda}(b)$. This condition is only satisfied in the most general case, when $\underline{\lambda}(a) = \underline{\lambda}(b) = \underline{0}$.

of $\underline{x}(t)$ and $\underline{\lambda}(t)$ still apply. But now equation (9) must also be satisfied; this requirement will lead to exactly r additional boundary relations. Because only those functions which satisfy equations (1), (6), and (7) are being considered, the variation $\delta\underline{v}(t)$ arises solely due to the variation of r boundary conditions, for some fixed set of boundary conditions on $\underline{x}(t)$ and $\underline{v}(t)$. The inhomogeneity $\underline{i}_S(t)$ is the same for the optimal and varied boundary conditions; hence, when equations (1), (6), and (7) are solved, $\delta\underline{v}(a)$ and $\delta\underline{v}(b)$ can both eventually be expressed as linear combinations of r arbitrary constants,

$$\underline{v}(a) = \underline{Py} , \qquad \underline{v}(b) = \underline{Qy} , \tag{10}$$

where P and Q are constant matrices, and $\underline{y}$ is an arbitrary column vector with r components. Using equation (10), (9) becomes

$$\underline{y}^T[\underline{Q}^T\underline{E}^T(b)\underline{v}(b) - \underline{P}^T\underline{E}^T(a)\underline{v}(a)] = 0. \tag{11}$$

Because the components of $\underline{y}$ are arbitrary and independent of one another,

$$\underline{Q}^T\underline{E}^T(b)\underline{v}(b) - \underline{P}^T\underline{E}^T(a)\underline{v}(a) = \underline{0} , \tag{12}$$

which is the desired set of r homogeneous boundary conditions on $\underline{v}(t)$.

When series inductances are also present, the situation is again more complicated. It is no longer possible to independently specify $\underline{x}(t)$ and $\underline{\lambda}(t)$, because at each port where such a series inductance is present, the state $\lambda_i(t)$ is an explicit function of $x_i(t)$ and the current source at that port. For each such port, the number of boundary conditions needed is reduced by one. An argument similar to the above shows that equation (8) will still provide exactly the required number of boundary conditions needed to specify a unique solution for equations (1), (6), and (7).

III. THE ADJOINT MATCHING NETWORK

The voltage $\underline{v}(t)$ and state $\underline{x}(t)$ which maximize the energy delivered to the load network N were obtained in section II. Now the load network N which produces these optimal functions when connected in parallel with the given network and excited by the current source $\underline{i}_S(t)$ can be found. When equation (2) is substituted in equation (6), the output equation of the matching network is obtained:

$$\underline{i}_N(t) = -\underline{B}^T(t)\underline{\lambda}(t) + [\underline{D}^T(t) - \dot{\underline{E}}^T(t)]\underline{v}(t) - \underline{E}^T(t)\dot{\underline{v}}(t). \qquad (13)$$

Hence, equation (7),

$$\dot{\underline{\lambda}}(t) = -\underline{A}^T(t)\underline{\lambda}(t) + \underline{C}^T(t)\underline{v}(t) \ , \qquad (14)$$

is the state equation of the matching network; the vector of Lagrange multipliers used in the variational process appears as the state vector of this matching network. Equations (13) and (14) are adjoint to equations (1) and (2); hence a necessary requirement for optimal matching is that the matching network satisfy differential equations which are adjoint to those satisfied by the given network. If it is demanded that equation (6) hold for aribtrary excitations, this adjoint requirement will in general lead to either negative or anticipative energy-storing elements in the matching network. If $\underline{\Phi}(t,\xi)$ satisfies

$$\frac{\partial}{\partial t}\underline{\Phi}(t,\xi) = \underline{A}(t)\underline{\Phi}(t,\xi) \qquad \underline{\Phi}(t,t) = \underline{I} \ , \qquad (15)$$

then the current in the given network, which is assumed to be non-anticipative, is given by[1]

[1] Zadeh and Desoer, op. cit., Chapter 6.

$$\underline{i}_1(t) = \underline{C}(t)[\underline{\Phi}(t,a)\underline{x}(a) + \int_a^t \underline{\Phi}(t,\xi)\underline{B}(\xi)\underline{v}(\xi)d\xi]$$

$$+ \underline{D}(t)\underline{v}(t) + \underline{E}(t)\dot{\underline{v}}(t) \; . \tag{16}$$

The zero state impulse response of this network is

$$\underline{y}_1(t,\tau) = \underline{C}(t)\underline{\Phi}(t,\tau)\underline{B}(\tau)u(t-\tau)$$

$$+ [\underline{D}(\tau) - \dot{\underline{E}}(\tau)]\delta(t-\tau) + \underline{E}(\tau)\delta'(t-\tau) \; . \tag{17}$$

The current in the non-anticipative matching network will be

$$\underline{i}_2(t) = -\underline{B}^T(t)[\underline{\Phi}^T(a,t)\underline{\lambda}(a) + \int_a^t \underline{\Phi}^T(\tau,t)\underline{C}^T(\tau)\underline{v}(\xi)d\xi]$$

$$+ [\underline{D}^T(t) - \dot{\underline{E}}^T(t)]\underline{v}(t) - \underline{E}^T(t)\dot{\underline{v}}(t) \; , \tag{18}$$

which corresponds to an impulse response

$$\underline{y}_2(t,\tau) = -\underline{B}^T(t)\underline{\Phi}^T(\tau,t)\underline{C}^T(\tau)u(t-\tau)$$

$$+ [\underline{D}^T(t) - \dot{\underline{E}}^T(t)]\delta(\tau-t) + \underline{E}^T(t)\delta'(\tau-t) \; . \tag{19}$$

The anticipative matching network will be described by

$$\underline{i}_3(t) = -\underline{B}^T(t)[\underline{\Phi}^T(b,t)\underline{\lambda}(b) - \int_t^b \underline{\Phi}^T(\xi,t)\underline{C}^T(\xi)\underline{v}(\xi)d\xi]$$

$$+ [\underline{D}^T(t) - \dot{\underline{E}}^T(t)]\underline{v}(t) - \underline{E}^T(t)\dot{\underline{v}}(t) \; , \tag{20}$$

$$\underline{y}_3(t,\tau) = \underline{B}^T(t)\underline{\Phi}^T(\tau,t)\underline{C}^T(\tau)u(\tau-t)$$

$$+ [\underline{D}^T(t) - \dot{\underline{E}}^T(t)]\delta(\tau-t) + \underline{E}^T(t)\delta'(\tau-t) \; . \tag{21}$$

The relation $\underline{y}_3(t,\tau) = \underline{y}_1(\tau,t)$ is the result obtained in previous investigations; this response is purely anticipative due to the presence of the step function $u(\tau-t)$. In some matching situations, when the given linear system can only be described by its impulse response, but not by differential state equations such as (1) and (2) (a so-called non-differential linear system), a purely anticipative matching system will be demanded. But, in the case of linear differential systems, the anticipative and non-anticipative matches (with an appropriate choice of initial conditions) are equivalent. This result is, of course, a consequence of the fact that both systems obey the same differential equations. The equivalence of (18) and (20) can also be shown using the group property of the state transition matrix. For the non-anticipative matching network,

$$\underline{\lambda}(t) = \underline{\Phi}^T(a,t)\underline{\lambda}(a) + \int_a^t \underline{\Phi}^T(\xi,t)\underline{C}^T(\xi)\underline{v}(\xi)d\xi \ . \tag{22}$$

Hence, when t = b,

$$\underline{\lambda}(b) = \underline{\Phi}^T(a,b)\underline{\lambda}(a) + \int_a^b \underline{\Phi}^T(\xi,b)\underline{C}^T(\xi)\underline{v}(\xi)d\xi \ . \tag{23}$$

Premultiplying by $\underline{\Phi}^T(b,t)$, and using the group property of the state transition matrix,

$$\underline{\Phi}^T(b,t)\underline{\lambda}(b) = \underline{\Phi}^T(a,t)\underline{\lambda}(a) + \int_a^b \underline{\Phi}^T(\xi,t)\underline{C}^T(\xi)\underline{v}(\xi)d\xi \ , \tag{24}$$

which can be rewritten as

$$\underline{\Phi}^T(b,t)\underline{\lambda}(b) - \int_t^b \underline{\Phi}^T(\xi,t)\underline{C}^T(\xi)\underline{v}(\xi)d\xi$$

$$= \underline{\Phi}^T(a,t)\underline{\lambda}(a) + \int_a^t \underline{\Phi}^T(\xi,t)\underline{C}^T(\xi)\underline{v}(\xi)d\xi \ . \tag{25}$$

This shows that equations (18) and (20) are equivalent, provided that the initial state $\underline{\lambda}(a)$ of the non-anticipative network is chosen consistently with the "initial" state $\underline{\lambda}(b)$ of the anticipative network. Of course, anticipation and negative energy storage are both non-physical demands, and optimal matching using physically realizable elements is impossible in the general case.

The matching network above must not only satisfy the adjoint differential equations; it must also supply the adjoint boundary conditions necessary for optimal matching. In most matching situations it is demanded that the boundary conditions be independent of the excitation, so that an optimal match is achieved for all signals (perhaps within some designated class of signals). In general, however, the boundary conditions depend on the excitation, even when anticipative matches are considered. "Initial" conditions for anticipative elements can only be specified after all unknown signals have ceased; but a system containing anticipative and non-anticipative elements will in general have response before excitation begins and after it ceases, and the boundary conditions will depend on the excitation. Here again, optimal matching in the general case imposed a non-physical demand on the matching network. However, when the excitation is not completely arbitrary, it may be possible to satisfy the requirement of adjoint boundary conditions independently of the excitation, in which case only the requirement of adjoint differential equations will remain to be satisfied.

IV. AVERAGE POWER MATCHING IN THE PERIODIC STEADY STATE

If a given network varies periodically with time, then its adjoint matching network, as determined in section III, also varies periodically, with the same period as the given network. When the excitation is also periodic, average power matching in the periodic steady state is of considerable interest. In order to determine the functions $\underline{v}(t)$ and $\underline{x}(t)$ which yield an optimal average power match, the integral in equation (3) is taken over one period. In addition, periodic boundary conditions are specified on $\underline{x}(t)$ and $\underline{v}(t)$ before the problem solution is undertaken:

$$\underline{x}(a) = \underline{x}(b) \qquad \underline{v}(a) = \underline{v}(b) \; . \tag{24}$$

Since the variation $\delta\underline{x}(t)$ in equation (8) is caused by variation of the boundary conditions on $\underline{x}(t)$, equation (24) demands that

$$\delta\underline{x}(a) = \delta\underline{x}(b) \qquad \delta\underline{v}(a) = \delta\underline{v}(b) \; . \tag{25}$$

Hence, in order to satisfy equation (8),

$$\underline{\lambda}(a) = \underline{\lambda}(b) \; , \tag{26}$$

which is the periodic adjoint boundary condition corresponding to the given condition (24). Since the excitation is also periodic, the boundary conditions (24) and (26) are those which are satisfied by the periodic part of the response of the matched system to the excitation $\underline{i}_S(t)$. The question of whether or not the system reaches the steady state starting from some arbitrary state has not arisen; optimal matching simply demands that the system be in the steady state at $t = a$. Once the system is in the steady state, however, it remains in the steady state, because the final conditions at t = b become the initial conditions for the next period, and so the solution there is the same periodic one. If the parallel combination of the given network and the adjoint matching network is stable, then the steady state will be reached regardless of the initial conditions on the networks, and optimal power matching is possible for arbitrary periodic excitations.

The preceeding results are now specialized to the time-invariant periodic steady state. In particular, it is shown that if $\underline{Y}(s)$ is the short circuit admittance matrix of a given linear time-invariant network, then the admittance matrices of both the anticipative and non-anticipative matching networks are given by

$$\underline{Y}_M(s) = \underline{Y}^T(-s) \ , \tag{27}$$

where a double sided Laplace transform is used to define the admittance matrix of the anticipative network. For time-invariant networks, the state transition matrix is[1]

$$\underline{\Phi}(t,\xi) = e^{\underline{A}(t-\xi)} \ . \tag{28}$$

Hence, $\underline{Y}(s)$ is given by

$$\underline{Y}(s) = \int_0^\infty e^{-st}[\underline{C}e^{\underline{A}t}\underline{B} + \underline{D}\delta(t) + \underline{E}\delta'(t)]dt \tag{29}$$

$$\underline{Y}(s) = \underline{C}[s\underline{I} - \underline{A}]^{-1}\ \underline{B} + \underline{D} + s\underline{E} \ , \tag{30}$$

which converges in the right half plane

$$\mathrm{Re}(s) > \sigma_c \ , \qquad \sigma_c = \max\{\mathrm{Re}\lambda_i\} \ , \tag{31}$$

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[1] Zadeh and Desoer, op. cit., page 2þ6.

where the λ_i are the eigenvalues of $\underline{A}$. For the anticipative matching network, the admittance matrix is defined by the two-sided Laplace transform

$$\underline{Y}_M(s) = \int_{-\infty}^{\infty} e^{-st}[\underline{B}^T e^{-\underline{A}^T t}\underline{C}^T u(-t) + \underline{D}^T\delta(t) - \underline{E}^T\delta'(t)]dt \ , \tag{32}$$

which, by changing t to -t in the integral, becomes

$$\underline{Y}_M(s) = \underline{B}^T[\int_0^{\infty} e^{st} e^{\underline{A}^T t} dt]\underline{C}^T + \underline{D}^T - s\underline{E}^T \ , \tag{33}$$

which satisfies equation (27). Here, the defining integral converges for $Re(s) < -\sigma_c$. The admittance matrix of the non-anticipative matching network is

$$\underline{Y}_M(s) = \int_0^{\infty} e^{-st}[-\underline{B}^T e^{-A^T t}\underline{C}^T + \underline{D}^T\delta(t) - \underline{E}^T\delta'(t)]dt \tag{34}$$

$$\underline{Y}_M(s) = -\underline{B}^T[s\underline{I} + \underline{A}]^{-1}\underline{C}^T + \underline{D}^T - s\underline{E}^T \ . \tag{35}$$

In equation (35), the integral converges for

$$Re(s) > \sigma_c' \quad , \quad \sigma_c' \ \ \max\{-Re\lambda_i\} \ . \tag{36}$$

Hence the matching network is specified by equation (27) in either case. However, the regions of convergence of the defining integrals are disjoint for the anticipative and non-anticipative matches; in fact, they are separated by a strip in the s plane which contains all the eigenvalues of the matrix $-\underline{A}$. Care must be taken when using the Laplace transform to solve differential equations involving the anticipative matching network because the Laplace transform for that network converges in a half plane open to the left.

V. SUFFICIENT CONDITIONS FOR AN OPTIMAL MATCH

Conditions which guarantee the existence of an optimal match can be obtained from the requirement that the second variation be strictly negative for non-vanishing variations. From the Appendix, the second variation is

$$\delta^2 e = \int_a^b \delta\underline{v}^T(t)[-\underline{C}(t)\delta\underline{x}(t) - \underline{D}(t)\delta\underline{v}(t) - \underline{E}(t)\delta\dot{\underline{v}}(t)]dt \; . \qquad (37)$$

Since $\underline{x}(t)$ is related to $\underline{v}(t)$ by equation (1), $\delta\underline{x}(t)$ can be found as a function of $\delta\underline{v}(t)$ and $\delta\underline{x}(a)$:

$$\delta\underline{x}(t) = \underline{\Phi}(t,a)\delta\underline{x}(a) + \int_a^t \underline{\Phi}(t,\xi)\underline{B}(\xi)\delta\underline{v}(\xi)d\xi \; . \qquad (38)$$

Hence the sufficient condition for the existence of an optimal match becomes

$$\int_a^b \delta\underline{v}^T(t)[\underline{C}\;(t)\underline{\Phi}(t,a)\delta\underline{x}(a) + \underline{C}\;(t)\int_a^t \underline{\Phi}(t,\xi)\underline{B}(\xi)\delta\underline{v}(\xi)d\xi$$
$$\underline{D}(t)\delta\underline{v}(t) + \underline{E}(t)\delta\dot{\underline{v}}(t)]dt > 0 \; . \qquad (39)$$

This expression is simply the energy absorbed by the given network during the interval [a,b] when its initial state is $\delta\underline{x}(a)$ and its excitation is $\delta\underline{v}(t)$. The variation $\delta\underline{v}(t)$ is a completely arbitrary function, subject only to certain continuity and differentiability requirements. In the matching situation where $\underline{x}(a)$ is specified in advance, equation (39) demands that the given network be strictly passive with respect to all voltage excitations (subject to the above conditions) starting in the zero state. In the most general matching situation, when $\underline{x}(a)$ is not specified, $\delta\underline{x}(a)$ is arbitrary, and equation (39) demands that the given network be strictly passive for all voltage excitations, starting in any state.

In the case of optimal average power matching in the periodic steady state, the boundary conditions which are specified on $\underline{x}(t)$ and $\underline{v}(t)$ before the problem solution is undertaken are $\underline{x}(a) = \underline{x}(b)$ and $\underline{v}(a) = \underline{v}(b)$. These conditions require that $\delta\underline{x}(a) = \delta\underline{x}(b)$ and $\delta\underline{v}(a) = \delta\underline{v}(b)$; hence equation (39) demands that the given network be strictly passive when operating in the steady state in response to arbitrary periodic excitation. This requirement is called the energy-absorbing condition.

VI. THE LOSSY AND LOSSLESS PARTS OF THE NETWORK RESPONSE

The preceeding results can be interpreted in terms of the lossy and lossless parts of the network response. Let the given network be excited by an arbitrary current source; then the energy delivered to the network will be

$$e = \int_a^b \underline{v}^T(t)\underline{i}(t)dt \ , \tag{40}$$

where $\underline{v}(t)$ is the voltage produced by the arbitrary current. The relation between $\underline{v}(t)$ and $\underline{i}(t)$ is

$$\underline{i}(t) = \underline{C}(t)[\underline{\Phi}(t,a)\underline{x}(a) + \int_a^t \underline{\Phi}(t,\xi)\underline{B}(\xi)\underline{v}(\xi)d\xi]$$
$$+ \underline{D}(t)\underline{v}(t) + \underline{E}(t)\dot{\underline{v}}(t) \ . \tag{41}$$

The current can be considered to consist of two parts; then the energy e can be written

$$e = \int_a^b \underline{v}^T(t)\underline{i}_1(t)dt + \int_a^b \underline{v}^T(t)\underline{i}_2(t)dt = e_1 + e_2 \ , \tag{42}$$

where

$$\underline{i}_1(t) = \tfrac{1}{2} \{\underline{C}(t)[\underline{\Phi}(t,a)\underline{x}(a) + \int_a^t \underline{\Phi}(t,\xi)\underline{B}(\xi)\underline{v}(\xi)d\xi]$$
$$- \underline{B}^T(t)[\underline{\Phi}^T(a,t)\underline{\lambda}(a) + \int_a^t \underline{\Phi}^T(\xi,t)\underline{C}^T(\xi)\underline{v}(\xi)d\xi]$$
$$+ [\underline{D}(t) + \underline{D}^T(t) - \dot{\underline{E}}^T(t)]\underline{v}(t) + [\underline{E}(t) - \underline{E}^T(t)]\dot{\underline{v}}(t)\} \tag{43}$$

$$\underline{i}_2(t) = \tfrac{1}{2}\,\{\underline{C}(t)[\underline{\Phi}(t,a)\underline{x}(a) + \int_a^t \underline{\Phi}(t,\xi)\underline{B}(\xi)\underline{v}(\xi)d\xi]$$

$$+ \underline{B}^T(t)[\underline{\Phi}^T(a,t)\underline{\lambda}(a) + \int_a^t \underline{\Phi}^T(\xi,t)\underline{C}^T(\xi)\underline{v}(\xi)d\xi$$

$$+ [\underline{D}(t) - \underline{D}^T(t) + \underline{\dot{E}}^T(t)]\underline{v}(t) + [\underline{E}(t) + \underline{E}^T(t)]\underline{\dot{v}}(t)\} \tag{44}$$

and $\underline{\lambda}(a)$ is not yet specified. Notice that $\underline{i}(t) = \underline{i}_1(t) + \underline{i}_2(t)$, and that $\underline{i}'(t) = \underline{i}_1(t) - \underline{i}_2(t)$ is the current in the adjoint matching network which would produce the same voltage $\underline{v}(t)$, when the initial state of that network is $\underline{\lambda}(a)$. The state of the given network at $t = b$ is

$$\underline{x}(b) = \underline{\Phi}(b,a)\underline{x}(a) + \int_a^b \underline{\Phi}(b,\xi)\underline{B}(\xi)\underline{v}(\xi)d\xi\ . \tag{45}$$

Also, the state of the adjoint network is

$$\underline{\lambda}(b) = \underline{\Phi}^T(a,b)\underline{\lambda}(a) + \int_a^b \underline{\Phi}^T(\xi,b)\underline{C}^T(\xi)\underline{v}(\xi)d\xi\ . \tag{46}$$

Using the separability of the state transition matrix, these equations can be written

$$\underline{X}^{-1}(b)\underline{x}(b) - \underline{X}^{-1}(a)\underline{x}(a) = \int_a^b \underline{X}^{-1}(\xi)\underline{B}(\xi)\underline{v}(\xi)d\xi\ , \tag{47}$$

$$\underline{X}^T(b)\underline{\lambda}(b) - \underline{X}^T(a)\underline{\lambda}(a) = \int_a^b \underline{X}^T(\xi)\underline{C}^T(\xi)\underline{v}(\xi)d\xi\ , \tag{48}$$

where $\underline{\Phi}(t,\xi)$ $\underline{X}(t)\underline{X}^{-1}(\xi)$. Using equation (41), the energy e_2 is

$$
\begin{aligned}
e_2 = \tfrac{1}{2} & \int_a^b \underline{v}^T(t)\underline{C}(t)\underline{\Phi}(t,a)\underline{x}(a)dt \\
& + \int_a^b \underline{v}^T(t)\underline{B}^T(t)\underline{\Phi}^T(a,t)\underline{\lambda}(a)dt \\
& + \int_a^b \underline{v}^T(t)\underline{C}(t) \int_a^t \underline{\Phi}(t,\xi)\underline{B}(\xi)\underline{v}(\xi)d\xi dt \\
& + \int_a^b \underline{v}^T(t)\underline{B}^T(t) \int_a^t \underline{\Phi}^T(\xi,t)\underline{C}^T(\xi)\underline{v}(\xi)d\xi dt \\
& + \int_a^b \underline{v}^T(t)[\underline{D}(t)\underline{v}(t) - \underline{D}^T(t)\underline{v}(t) + \dot{\underline{E}}^T(t)\underline{v}(t) \\
& \qquad\qquad + \underline{E}(t)\dot{\underline{v}}(t) + \underline{E}^T(t)\dot{\underline{v}}(t)]dt \ . \qquad (49)
\end{aligned}
$$

Upon an interchange of the order of integration in the second double integral and the substitution of equations (47) and (48),

$$e_2 = \tfrac{1}{2} \left[\underline{x}^T(t)\underline{\lambda}(t) + \underline{v}^T(t)\underline{E}(t)\underline{v}(t)\right]_a^b \ . \qquad (50)$$

Hence $\underline{i}_2(t)$ can be regarded as the lossless part of the network response, in the sense that any signal which takes the network from the zero state at t = a back to the zero state at t = b delivers no energy to this part of the network response; all of the energy delivered to the network by such signals is delivered to the lossy part of the network response determined by $\underline{i}_1(t)$. This relation is true regardless of the choice of $\underline{\lambda}(a)$ originally made in equations (43) and (44).

The requirements of optimal matching are now clear. The optimal matching network is that network whose lossy part is the same as the lossy part of the given network, and whose lossless part is the negative of the lossless part of the given network.

Consider again the simplified case in which there are no series inductances or shunt capacitances at the ports of the given network. In the most general case of optimal matching, $\underline{\lambda}(a) = \underline{\lambda}(b) = \underline{0}$, and the energy e_2 is zero. The excitation supplies no energy to the lossless part of the given network; and, because the lossless parts of the given and matching networks are negatives of one another, the excitation supplies no energy to the lossless part of the matching network. In the case where $\underline{x}(a)$ is specified in advance, and $\underline{\lambda}(b)$ is required to be zero to satisfy the boundary conditions, $e_2 = -\frac{1}{2}\underline{x}^T(a)\underline{\lambda}(a)$. That is, the lossless part of the given network loses an amount of energy e_2 during the matching interval; this same amount of energy is gained by the lossless part of the matching network. These requirements on the lossless part of the matching network give rise to the demand that the matching network contain either negative or anticipative elements, except perhaps in the periodic steady state, where a certain average stored energy is established in each element in the process of reaching the steady state. The function $[\underline{x}^T(t)\underline{\lambda}(t) + \underline{v}^T(t)\underline{E}(t)\underline{v}(t)]/2$, which is in fact the conjunct of the adjoint equations which describe the two networks, is intimately related to the stored energy of the system, and the exact nature of this relation merits further investigation.

Finally, equation (39) can be interpreted as requiring that the lossy part of the given network (and hence of the matching network also) be strictly lossy, that is, that energy be absorbed for arbitrary signals.

In investigations of the stability of time-varying networks[1], it has been observed in simple cases that the requirement for a network to be stable is precisely the same as the requirement that an optimal match exist. It may be conjectured that if the lossy part of a network's response is strictly lossy, and if the network can store only positive energy, then the network is stable. It is not attempted to prove this assertion here.

[1] Rohrer, R. A., "Energy Function Analysis of Time-Varying Networks", Ph.D. Dissertation, University of California, Berkeley, May 1963.

VII. CONCLUSION

It has been found that optimal matching of linear networks requires the use of a network which is adjoint to the given network. This requirement of adjointness applies not only to the network differential equations, but also to the boundary conditions satisfied by its state and voltage vectors. In general, the adjoint matching network is not physically realizable because it is required to be anticipative or to store negative energy. It has been suggested that an optimal match may be realizable in the periodic steady state; this situation is of practical importance, and further investigation of the matter is needed. The matching procedure has been interpreted in terms of the lossy and lossless parts of the network response; the energy supplied to the lossless part of the network response appears to be intimately related to the stored energy of the matched system. Finally, it has been suggested that the condition of stability of the given network is the same as the condition that an optimal match exist.

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APPENDIX

A. Necessary Conditions

From section II of the text, the functional which is to be maximized is

$$e = \int_a^b \underline{v}^T(t)[\underline{i}_S(t) - \underline{C}(t)\underline{x}(t) - \underline{D}(t)\underline{v}(t) - \underline{E}(t)\dot{\underline{v}}(t)]dt \ , \tag{51}$$

under the constraint

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{B}(t)\underline{v}(t) \ . \tag{52}$$

According to the method of Euler and Lagrange, the necessary conditions for the functional (51) to have an extremum under the constraint relation (52) are the same as the necessary conditions for the following functional to have an extremum, with no side constraints:

$$e' = \int_a^b \begin{array}{l} \underline{v}^T(t)[\underline{i}_S(t) - \underline{C}(t)\underline{x}(t) - \underline{D}(t)\underline{v}(t) - \underline{E}(t)\dot{\underline{v}}(t)] \\ - \underline{\lambda}^T(t)[\dot{\underline{x}}(t) - \underline{A}(t)\underline{x}(t) - \underline{B}(t)\underline{v}(t)] \end{array} dt \ , \tag{53}$$

where $\underline{\lambda}(t)$ is an (as yet) unknown function. When the function $\underline{x}(t)$ is varied, the first variation of e' is

$$\delta e' = \int_a^b [\underline{C}(t)\delta\underline{x}(t) + \underline{\lambda}^T(t)\delta\dot{\underline{x}}(t) - \underline{\lambda}^T(t)\underline{A}(t)\delta\underline{x}(t)]dt \ . \tag{54}$$

Upon integrating the term $\underline{\lambda}^T(t)\delta\dot{\underline{x}}(t)$ by parts and transposing all the terms, the first variation becomes

$$e' = \int_a^b \delta\underline{x}^T(t)[-\dot{\underline{\lambda}}(t) - \underline{A}^T(t)\underline{\lambda}(t) + \underline{C}^T(t)\underline{v}(t)]dt + [\delta\underline{x}^T(t)\underline{\lambda}(t)]_a^b \ . \tag{55}$$

If an optimal is to exist at all, the above expression, which is linear in the variation $\delta\underline{x}(t)$, must vanish. The integral must vanish whether the end points are fixed or not[1]; since the variation $\delta\underline{x}(t)$ is an arbitrary (continuous and differentiable) function, this requires that

$$\dot{\underline{\lambda}}(t) = -\underline{A}^T(t)\lambda(t) + \underline{C}^T(t)\underline{v}(t) , \tag{56}$$

which is one of the two necessary Euler equations. If the end points of $\underline{x}(t)$ are fixed before the problem solution is undertaken, then the conjunct which arises when terms containing $\delta\underline{x}(t)$ are integrated by parts vanishes automatically. When some or all of the end points are not specified in advance, the requirement that the conjunct vanish,

$$[\underline{x}^T(t)\underline{\lambda}(t)]_a^b = 0 , \tag{57}$$

provides the additional information needed to determine the optimal function $\underline{x}(t)$. When the function $\underline{v}(t)$ is varied, the first variation of the functional e' is

$$\begin{aligned} e' = \int_a^b \{ &-\delta\underline{v}^T(t)[\underline{i}_S(t) - \underline{C}(t)\underline{x}(t) - \underline{D}(t)\underline{v}(t) - \underline{E}(t)\dot{\underline{v}}(t)] \\ &+ \underline{v}^T(t)\underline{D}(t)\delta\underline{v}(t) + \underline{v}^T(t)\underline{E}(t)\delta\dot{\underline{v}}(t) - \underline{\lambda}^T(t)\underline{B}(t)\delta\underline{v}(t)\}dt. \end{aligned} \tag{58}$$

After transposing the terms which contain $\underline{v}^T(t)$ and integrating by parts the term containing $\delta\dot{\underline{v}}(t)$, this becomes

$$\begin{aligned} e' = \int_a^b \delta\underline{v}^T(t)[&\underline{i}_S(t) - \underline{C}(t)\underline{x}(t) - \underline{D}(t)\underline{v}(t) - \underline{E}(t)\dot{\underline{v}}(t) \\ &- \underline{D}^T(t)\underline{v}(t) + \dot{\underline{E}}^T(t)\underline{v}(t) + \underline{E}^T(t)\dot{\underline{v}}(t) \\ &+ \underline{B}^T(t)\underline{\lambda}(t)]dt + [\delta\underline{v}^T(t)\underline{E}^T(t)\underline{v}(t)]_a^b . \end{aligned} \tag{59}$$

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[1] Elsgolc, L. E., Calculus of Variations, Addison-Wesley, Reading, Mass., 1962.

The requirement that the first variation vanish leads to

$$\underline{i}_S(t) - \underline{C}(t)\underline{x}(t) - \underline{D}(t)\underline{v}(t) - \underline{E}(t)\dot{\underline{v}}(t)$$

$$+ \underline{B}^T(t)\underline{\lambda}(t) - [\underline{D}^T(t) - \dot{\underline{E}}^T(t)]\underline{v}(t) + \underline{E}^T(t)\dot{\underline{v}}(t) = \underline{0} \ , \qquad (60)$$

$$[\delta\underline{v}^T(t)\underline{E}^T(t)\underline{v}(t)]_a^b = 0 \ . \qquad (61)$$

Equation (60) is the second necessary Euler equation; equation (61) provides information which determines the end points if they are not originally specified in the problem.

B. Sufficient Conditions

The method of Euler and Lagrange simplifies the task of determining the necessary Euler equations. However, the requirements of the sufficient conditions for the existence of a maximum for the value of e are most easily interpreted if the equation of constraint (52) is solved for $\underline{x}(t)$; then $\underline{x}(t)$ can be eliminated from the functional e, and only $\underline{v}(t)$ need be varied. If this procedure is carried out, the second variation of e becomes

$$\delta^2 e = \int_a^b -\delta\underline{v}^T(t)[\underline{C}(t)\underline{\Phi}(t,a)\delta\underline{x}(a) + \underline{C}(t)\int_a^t \underline{\Phi}(t,\xi)\underline{B}(\xi)\delta\underline{v}(\xi)d\xi$$

$$+ \underline{D}(t)\delta\underline{v}(t) + \underline{E}(t)\delta\dot{\underline{v}}(t)]dt \ , \qquad (62)$$

which is the total variation of e if all the preceeding necessary conditions are satisfied by the optimal functions $\underline{x}(t)$ and $\underline{v}(t)$. Hence a sufficient condition for the existence of a maximum for the value of e is that the above second variation be strictly negative for all non-vanishing variations $\delta\underline{x}(a)$ and $\delta\underline{v}(t)$.

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Los Angeles 45, California

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The calculus of variations is used to obtain the optimal load network for a linear time-varying n-port network excited at each port by arbitrary signals. The given network is represented using a state variable description; use of a matrix of Lagrange multipliers in the variational process then leads directly to the state variable description of the adjoint matching network. The choice of anticipative or non-anticipative load networks is considered. The general result is then specialized to obtain the usual condition required for matching time-invariant networks in the periodic steady state. Conditions which guarantee the existence of an optimal match are obtained from the second variation. Finally, the entire matching process is interpreted in terms of the lossy and lossless parts of the network response.

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